Exam Lie Groups in Physics

Date	January 30, 2020
Room	NB 5113.0201
Time	08:30 - 11:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	6	2a)	7	3a)	7	4a)	7
1b)	6	2b)	8	3b)	8	4b)	8
1c)	6	2c)	7	3c)	7		
1d)	6	2d)	7				

$$\text{Result } = \frac{\sum \text{points}}{10} + 1$$

Problem 1

(a) Provide the definition of a Lie algebra.

(b) Provide the definitions of an invariant subalgebra and a simple Lie algebra.

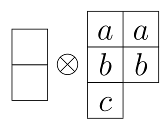
(c) Give an example of a Lie group encountered in physics that is not simple.

(d) If the coset space G/H forms a group, how are the Lie algebras of G and H related?

Problem 2

Consider the Lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

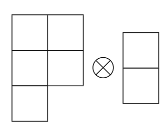
(a) Consider the following direct product of irreps of the Lie algebra $su(n)$:



Write down all allowed sequences ("words") consisting of the letters a, a, b, b, c.

(b) Decompose the above direct product of irreps into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(c) Do the same direct product but now in reverse order:



(d) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(3)$ and $su(4)$. Indicate the complex conjugate and inequivalent irreps whenever appropriate.

Problem 3
Consider the group $O(3)$ of real orthogonal 3×3 matrices.

(a) Show the isomorphism $O(3) \cong \mathsf{Z}_2 \otimes SO(3)$.

(b) Show that the symmetric and antisymmetric tensors $x_i y_j \pm x_j y_i$ do not mix under $O(3)$ transformations.

(c) Argue that the defining representation of $O(3)$ is irreducible and becomes reducible when restricting to an $O(2)$ subgroup.

Problem 4
Consider the four-dimensional representation of the generators of the Lorentz group:

$$(M^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha} g^{\nu}{}_{\beta} - g^{\nu\alpha} g^{\mu}{}_{\beta})$$

(a) Write down the matrices for the following two cases: $\mu = 0, \nu = 2$ and $\mu = 1, \nu = 2$.

(b) Derive an expression for $\exp(-i\chi M^{02})$ in terms of hyperbolic cosines and sines, using the expression for M^{02} obtained in part (a). Conclude which Lorentz transformation it corresponds to. Recall that $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.